doi:10.1006/jabr.2001.8841, available online at http://www.idealibrary.com on IDEAL®

Splitting Comodules over Hopf Algebras and Application to Representation Theory of Quantum Groups of Type $A_{0|0}$

Phùng Hồ Hải

Hanoi Institute of Mathematics, P.O. Box 631, 10000 Boho, Hanoi, Vietnam
E-mail: phung@thevinh.ncst.ac.vn

Communicated by Susan Montgomery

Received November 22, 1999

In this work we study some properties of comodules over Hopf algebras possessing integrals (co-Frobenius Hopf algebras). In particular we give a necessary and sufficient condition for a simple comodule to be injective. We apply the result obtained to the classification of representations of quantum groups of type $A_{0|0}$.

INTRODUCTION

The notion of integrals on Hopf algebras is motivated by the Haar integral on compact groups. In fact, the axiom of the Haar integral on a compact group can be given in a purely algebraic way as a linear functional on the algebra of representative functions on the group (which is a Hopf algebra), satisfying an invariant property, which can be explained in terms of the coproduct on the (Hopf) algebra of functions. One takes this axiom for the definition of an integral on an arbitrary Hopf algebra over an arbitrary field.

Since the pioneering work of Sweedler [22], integrals on Hopf algebras were studied by several authors, e.g., in [1, 4, 5, 19–21]. Among others, Sweedler proved the existence and uniqueness up to a constant of a (non-zero) integral on any finite-dimensional Hopf algebra. A theorem of Sullivan states that if a non-zero integral exists on a Hopf algebra then it is uniquely determined up to a constant. We shall also assume that "integral" means "non-zero integral."

0021-8693/01 $35.00

In representation theory of compact groups, one uses the Haar integral to deduce the semisimplicity of representations. There is an analogue in comodule theory of Hopf algebras. If a Hopf algebra possesses an integral which does not vanish at the unit element, then it is cosemisimple and conversely, if all its comodules are semisimple then it possesses an integral which does not vanish at the unit element. However, the integral may vanish at the unit element, which is equivalent to the non-cosemisimplicity of the Hopf algebra. While there are many examples of non-cosemisimple, finite-dimensional Hopf algebras, not so many infinite-dimensional, non-cosemisimple Hopf algebras with integrals are known. Moreover, a theorem of Sullivan [21] states that a commutative Hopf algebra over a field of characteristic zero possesses an integral if and only if it is cosemisimple.

Some new examples of infinite-dimensional, non-cosemisimple Hopf algebras with integrals come from Lie supergroups and quantum (super) group theory. In studying Haar measure on compact supergroups, Berezin found a remarkable fact that the Haar measure exists but the whole volume of the supergroup with respect to this measure may vanish (see, e.g., [3]). In other words, the function algebra on a compact supergroup is an infinite-dimensional Hopf superalgebra with integral, which may be non-cosemisimple.

In [2] some new examples of non-cosemisimple Hopf algebras with integral are constructed.

In [9] the author showed that the Hopf algebras associated with certain (non-even) Hecke symmetries (i.e., quantum groups of type A) were non-cosemisimple, infinite-dimensional Hopf algebras with integrals.

In studying representations of simple Lie superalgebras of classical type, Kac found out that their irreducible representations splitted into two classes of typical and atypical representations (see, e.g., [13]). It turns out that there is an analogous notion for simple comodules over a Hopf algebra with integral and the integral provides a necessary and sufficient condition for a simple comodule to be “typical” (called “splitting” in this work). This is the main result of the first part of this work. In the second part we apply this result to studying representations of quantum groups of type $A_{0|0}$, i.e., Hopf algebras associated to Hecke symmetries of birank $(1, 1)$. We classify irreducible representations of these quantum groups and using the classification result we are also able to classify the symmetries.

The work is briefly divided into two parts. In Section 1 we recall some definitions and known facts of integrals on Hopf algebras. Then we recall in Section 2 the convolution product, which is given in terms of the integral, making the Hopf algebra into a non-unital associated algebra. Using this construction, we prove a key result in Theorem 2.3. In Section 3 we introduce the notion of splitting comodules, which means injective, projective simple comodules. We provide in Theorem 3.1 a necessary and sufficient condition for a simple comodule to be a splitting comodule.

In the second part of the work, Section 4, we apply the result of the first part to Hopf algebras associated to Hecke symmetries of birank $(1,1)$, i.e., Hecke symmetries, whose associated quantum exterior algebras have the Poincaré series equal to $(1+at)(1-bt)^{-1}$, $a,b>0$. We show that simple comodules of these Hopf algebras can be labeled by pairs of integers (k,l), where $(0,0)$ is the trivial comodule, $(1,0)$ is the fundamental comodule, $(-1,0)$ is its dual, and the comodule labeled by (k,l) is splitting iff $k+l\neq 0$. We show that the dimension of a simple comodule is 2 or 1 depending on whether it is splitting or not. Using this we classify the Hecke symmetries of birank $(1,1)$. It turns out that a Hecke symmetry of birank $(1,1)$ should be defined on a vector space of dimension 2. According to the classification result of Hietarinta [12], there are no other than those found by Manin [18] and Takeuchi and Tambara [23].

1. CO-FROBENIUS COALGEBRAS AND HOPF ALGEBRAS

We work over a field k. Every tensor product if not explicitly indicated means tensor product over k.

Let C be a coalgebra and M be a right C-comodule, the coaction of C on M is denoted by ρ, ρ: $M\to M\otimes C$, $\rho(v)=v_0\otimes v_1$. Let $C^*:=\mathrm{Hom}_k(C,k)$ be the dual of C. Then C^* is an algebra, acting on M from the left in the following way $\phi\rightharpoonup v:=v_0\phi(v_1)$. Analogously, if λ: $N\to C\otimes N$, $\lambda(v)=v_{-1}\otimes v_0$ is a left C-comodule, then it is a right C^*-module through the action $v\leftharpoonup\phi:=\phi(v_{-1})v_0$.

Thus, we have a functor from the category of right (left) C-comodules into the category of left (right) C^*-modules, which can be easily seen to be fully faithful and exact. A C^*-module may not be a C-comodule by the above correspondence. C^*-modules induced from C-comodules are called rational modules. Each left C^*-module M contains a unique maximal rational submodule, denoted by ${}_{\mathrm{rat}}M$. Analogously, for a right C^*-module M, its rational submodule will be denoted by M_{rat}.

Let M be a C-comodule. The map ρ: $M\to M\otimes C$ induces a map $M^*\otimes M\to C$, which can be considered as a coalgebra homomorphism or as a morphism of C-comodules, where C coacts on $M^*\otimes M$ on the second tensor component. In the latter case, we shall use the notation $(M^*)\otimes M$ to indicate that C coacts only on M. The image of $M^*\otimes M$ in C is called the coefficient space of M, denoted by $\mathscr{C}f(M)$.

Let M be a simple (left or right) C-comodule. The fundamental theorem of comodules (saying that a finitely generated comodule is finite-dimensional) implies that M is finite-dimensional. Let $\mathscr{D}:=\mathrm{End}^C(M)$.

Then, by Schur's lemma, $\mathcal{D}$ is a division algebra over k and M is a vector space over $\mathcal{D}$. We have $\mathcal{C}f(M) \cong M^* \otimes_{\mathcal{D}} M$, as coalgebras [6]. Let $\{M_\alpha | \alpha \in \mathcal{A}\}$ be the set of all isomorphism classes of simple C-comodules. We define $\mathcal{D}_\alpha := \operatorname{End}^C(M_\alpha)$ and $m_\alpha := \dim_{\mathcal{D}_\alpha}(M_\alpha)$, $d_\alpha := \sqrt{\dim_k \mathcal{D}_\alpha}$. Note that d_α are positive integers and if k is algebraically closed then $\mathcal{D}_\alpha = k$, $\forall \alpha$; i.e., $d_\alpha = 1$.

By definition, the socle of a comodule M is the sum of all its simple subcomodules. The sum is direct and is denoted by $\sigma(M)$. The injective envelope (or hull) of M is by definition an injective comodule $\mathcal{J}(M)$ together with a morphism $M \to \mathcal{J}(M)$ inducing an isomorphism $\sigma(M) \to \sigma(\mathcal{J}(M))$. It is easy to see that the injective envelope of a simple comodule, if it exists, is indecomposable. The following results can be found in [6].

1.1.

Let C be a coalgebra. Consider comodules over C.

(i) The injective envelope of any comodule exists uniquely.

(ii) C itself decomposes into indecomposable injective subcomodules as follows

$$C \cong \bigoplus_{\alpha \in \mathcal{A}} \mathcal{J}(M_\alpha)^{\oplus m_\alpha}. \tag{1}$$

(iii) If $C = \oplus_{\lambda \in \mathcal{L}} N_\lambda$ is another decomposition of the same type then for each $\alpha \in \mathcal{A}$, the set $\{\lambda \in \mathcal{L} | N_\lambda \cong \mathcal{J}(M_\alpha)\}$ contains exactly m_α elements.

A bilinear form b on C is called balanced if, for all $\phi \in C^*$,

$$b(x \leftharpoonup \phi, y) = b(x, \phi \rightharpoonup y).$$

Balanced bilinear forms on C are in 1–1 correspondence with right C^*-modules homomorphism r: $C \to C^*$ by the formula $r(x)(y) = b(x, y)$ and in 1–1 correspondence with left C^*-modules homomorphism r: $C \to C^*$ by the formula $l(x)(y) = b(y, x)$.

A coalgebra is called left (right) co-Frobenius if there exists a left (right) monomorphism of C^*-modules $C \to C^*$. The following result is due to Lin [14].

1.2.

If C is a *left* co-Frobenius coalgebra then:

(i) The injective cover of every finite dimensional *right* C-comodule is finite-dimensional.

(ii) Every injective *right* C-comodule is projective.

Let now H be a Hopf algebra. Then k is a left (right) H-comodule by means of the unit map. A left (right) integral on H is a non-zero H-comodule morphism $H \to k$, where H is considered as left (right) comodule on itself by means of the coproduct. Let $\int_r$ (resp. $\int_l$) denote a left (resp. right) integral on H, then we have

$$a_1 \int_l (a_2) = \int_l (a), \tag{2}$$

$$\int_r (a_1) a_2 = \int_r (a), \tag{3}$$

$\forall\, a \in H$.

We need the following information on integrals [14].

1.3.

Let H be a Hopf algebra. The following conditions are equivalent.

(i) H possesses a left or a right integral.

(ii) H is left or right co-Frobenius as a coalgebra.

(iii) The injective envelope of a simple left or right comodule (hence of any simple left or right comodules) is finite-dimensional.

Define bilinear form b: $b(x, y) := \int_l (xS(y))$. Using the identity

$$h_1 \int_l (h_2 S(g)) = \int_l (hS(g_1)) g_2 \tag{4}$$

which follows immediately from the definition of $\int_l$, we can easily show that b is balanced. The following result is due to Ştefan [20].

1.4.

Let H be a Hopf algebra with integral. Then the following facts hold.

(i) The bilinear form b is non-degenerate.

(ii) For any finite-dimensional H-comodule, $\dim_k(\mathrm{Hom}^H(H, M)) = \dim_k M$.

In particular, we have

(i) the antipode is injective, and

(ii) there exists h such that $\int_l (S(h)) \neq 0$. Since $\int_l \circ S$ satisfies (3), it is a right integral on H.

In fact, we have a stronger statement for the antipode which is due to Radford [19].

1.5.

Let H be a Hopf algebra with integral. Then the antipode is bijective.

Assume for a moment that the field k is algebraically closed. Let R be the coradical of H; i.e., $R = \bigoplus_\alpha \mathscr{C}f(M_\alpha)$, where $\{M_\alpha, \alpha \in \mathscr{A}\}$ is the set of all simple left (or right) H-comodules. As we have seen in the previous section, $\mathscr{C}f(M_\alpha) \cong M(m_\alpha)^*$, where $M(m_\alpha)$ is the matrix ring of degree m_α. $m_\alpha := \dim_k(M_\alpha)$. Fix idempotents $\{e_{\alpha,i} | \alpha \in \mathscr{A}, 1 \le i \le m_\alpha\}$ of the algebra $M(m_\alpha)$. They can be considered as linear functionals on R by defining

$$e_{\alpha,i}(\mathscr{C}f(M_\beta)) = 0, \text{ whenever } \alpha \neq \beta.$$

A theorem of Sweedler–Sullivan [21], stating that there exists a coalgebra projection $H \to R$, implies that $e_{\alpha,i}$ can be extended on the whole H and that

$$H \leftharpoonup e_{\alpha,i} \quad (\text{resp. } e_{\alpha,i} \rightharpoonup H) \text{ is a right (resp. left) } H\text{-comodule.}$$

Consequently, we have a decomposition

$$H \cong \bigoplus_{\substack{\alpha \in \mathscr{A}, \\ 1 \le i \le m_\alpha}} H \leftharpoonup e_{\alpha,i}$$

of right H-comodules and an analogous one for left H-comodules.

On the other hand, it is easy to see that $M_\alpha \subset H \leftharpoonup e_{\alpha,i}$ as right H-comodules. Thus, comparing with the decomposition in 1.1, we have:

1.6.

Assume that the field k is algebraically closed. Let $e_{\alpha,i}$ be linear functionals given as above. Then

$$\mathscr{J}(M_\alpha) \cong H \leftharpoonup e_{\alpha,i} \tag{5}$$

as right H-comodules.

Let M be a finite-dimensional right H-comodule then $M^* := \operatorname{Hom}_k(M, k)$ is also a right comodule with the coaction given by the equation

$$\rho(\phi)(x) := \phi_0(x)\phi_1 = \phi(x_0)S(x_1), \ x \in M, \phi \in M^*.$$

The map ev: $M^* \otimes M \to k$, $\phi \otimes x \mapsto \phi(x)$ is a morphism of H-comodules. The pair (M^*, ev) is called a left dual to M; it is defined uniquely up to isomorphisms. There exists a monomorphism db: $k \to M \otimes M^*$, defined by the conditions $(\mathrm{ev} \otimes \mathrm{id}_{M^*})(\mathrm{id}_{M^*} \otimes \mathrm{db}) = \mathrm{id}_{M^*}$ and $(\mathrm{id}_M \otimes \mathrm{ev})(\mathrm{db} \otimes \mathrm{id}_M) = \mathrm{id}_M$, which is also a comodule morphism. The notion of right dual is defined analogously; for instance, (M, ev) is the right dual to M^*.

Thus, we see that the left dual to a finite-dimensional comodule always exists. If the antipode is bijective then the right dual to any finite-dimensional comodule also exists. We shall need the following isomorphism, given by manipulating the morphism ev and db: for any finite-dimensional comodule N,

$$\mathrm{Hom}^H(M \otimes N, P) \cong \mathrm{Hom}^H(M, P \otimes N^*) \tag{6}$$

$$\mathrm{Hom}^H(M, N \otimes P) \cong \mathrm{Hom}^H(N^* \otimes M, P). \tag{7}$$

LEMMA 1.7. *Let M be a finite-dimensional comodule over a coalgebra C. Then we have*:

(i) *M is projective* (*resp. injective*) *if and only if it is projective* (*resp. injective*) *in the category of finite-dimensional C-comodules*;

(ii) *If M^* is projective* (*resp. injective*) *then M is injective* (*resp. projective*);

(iii) *If the antipode is bijective, then M is injective* (*resp. projective*) *iff M^* is projective* (*resp. injective*).

Proof. (i) The proof is a routine application of Zorn's lemma and is left to the reader.

(ii) Eqs. (6) and (7) imply

$$\mathrm{Hom}^H(M^*, N^*) \cong \mathrm{Hom}^H(N, M). \tag{8}$$

Thus, if M^* is projective (resp. injective) then M is injective (resp. projective) in the category of finite-dimensional comodules. By (i), M is then injective (resp. projective). (ii) is thus proved.

Assume that the antipode is bijective then we can also define right dual to any finite-dimensional comodules. Since M is the right dual to M^*, (iii) follows from a similar discussion. ∎

2. INJECTIVE ENVELOPE AND PROJECTIVE COVER

We have seen in the previous section that the injective envelope $\mathcal{J}(M)$ of a simple comodule M over a Hopf algebra with integral H is indecomposable and projective; hence so is $\mathcal{J}(M)^*$, the dual comodule. A natural question arises, how the socle of $\mathcal{J}(M)^*$ relates to M. This section is devoted to this question. We show in Corollary 2.4 that the socle of $\mathcal{J}(M)^*$ is isomorphic to the tensor product of M^{***} with a distinguished one-dimensional comodule, introduced by Radford [19]. To do this, we use the convolution product on H, given in terms of the integral H, equipped with this convolution product, is isomorphic to ${}_{\mathrm{rat}}H^*$ as non-unital algebras. The results preceeding Theorem 2.3 are known. We recall them in a slightly different way for the sake of our convenience [4, 19, 24].

Define the following binary operation on H.

$$g * h := h_1 \int_l (h_2 S(g)) = \int_l (hS(g_1))g_2 \text{ (by (4))}.$$

Using (4) we can easily check that $*$ is associative. $*$ is called convolution product on H. Denote $\check{H} := (H, *)$. Then $\check{H}$ is a (non-unital) algebra. In fact, $\check{H}$ is isomorphic to ${}_{\mathrm{rat}}(H^*)$ as non-unital algebras through the Sweedler map $h \to \int^h : g \mapsto \int_l (gS(h))$.

Let V be a right H-comodule. Then V is a left $\check{H}$-module by means of the action

$$h * v := v_0 \int_l (v_1 S(h)).$$

The verification again uses (4). Denote $\check{V} := (V, *)$.

Let $f\colon V \to W$ be a homomorphism of right H-comodules; i.e., $f(v)_0 \otimes f(v)_1 = f(v_0) \otimes v_1$. We have

$$h * f(v) = f(v)_0 \int_l (f(v)_1 S(h)) = f(v_0) \int_l (v_1 S(h)) = f(h * v).$$

Thus f is a homomorphism of left $\check{H}$-modules. Conversely, if f is a homomorphism $\check{V} \to \check{W}$, then we have, for all $h \in H$,

$$f(v)_0 \int_l (f(v)_1 S(h)) = f(v_0) \int_l (v_1 S(h)).$$

By the non-degeneracy of the integral (1.4, (i)), we have

$$f(v)_0 \otimes f(v)_1 = f(v_0) \otimes v_1,$$

which means that f is a homomorphism of right H-comodules. Thus we have

$$\mathrm{Hom}_{\check{H}}(\check{V}, \check{W}) = \mathrm{Hom}^H(V, W). \tag{9}$$

An $\check{H}$-module V is called unital if for any element $v \in V$, there exists (not uniquely) an element $h \in H$, such that $h * v = v$. For example, if V is an H-comodule then $\check{V}$ is unital. Indeed, let $v \in V$. $h * v = v$ means $v_0 \int_l (v_1 S(h)) = v$. From the non-degeneracy of the integral, we can choose h such that $\int_l (v_1 S(h)) = \varepsilon(v_1)$. Whence we get $h * v = v$.

Let V now be a unital H-module. We define a coaction of H on V as follows. Let $h \in \check{H}$ be such that $v = h * v$. Set $\hat{\delta}(v) := h_1 * v \otimes h_2$. We show that $\hat{\delta}$ is independent of the choice of h, and that it is in fact a coaction of H on V. The independence of the choice of h means that, whenever $h * v = 0$, we have $h_1 * v \otimes h_2 = 0$. Indeed, we have

$$\int_l (h_2 S(g)) h_1 * v = (g * h) * \bar{v} = g * (h * v) = 0,$$

for all $g \in H$. By the non-degeneracy of $\int_l$, we conclude that $h_1 * v \otimes h_2 = 0$. The coassociativity and counital of $\hat{\delta}$ are also checked directly using (4). Moreover, denoting by $\widehat{V}$ the resulting H-comodule, we also have $\check{\widehat{V}} \cong V$.

Thus we have defined a functor $\hat{}$ from the category of unital $\check{H}$-modules to the category of H-comodules, which is inverse to $\check{}$. It is obvious that unital $\check{H}$-modules form a full subcategory of the category of $\check{H}$-modules.

Proposition 2.1 [4, Corollary 3.6]. *The functors* $\check{}$ *and* $\hat{}$ *are inverse to each other. Hence the category of* H*-comodules and the category of unital* $\check{H}$*-modules are equivalent as abelian categories.*

We also need another action of $\check{H}$ on a right comodule V of H, given by

$$h \circ v := v_0 \int_l (hS(v_1)).$$

The following calculation shows that $\circ$ is an action:

$$\begin{aligned} g \circ (h \circ v) &= g \circ v_0 \int_l (hS(v_1)) = v_0 \int_l (gS(v_1)) \int_l (hS(v_2)) \\ &= v_0 \int_l \Big(hS\Big(\int_l (gS(v_1)) v_2 \Big) \Big) = v_0 \int_l (hS(g_1)) \int_l (g_2 S(v_1)) \\ &= (g * h) \circ v. \end{aligned}$$

It is again easy to check that $\check{V} := (V, \circ)$ is a left $\check{H}$-module.

Thus we have defined another functor from the category of H-comodules to the category of unital $\check{H}$-comodules. A natural question is whether this functor has an inverse. Let V be a unital $\check{H}$-module. We are thus looking for a coaction $\delta^{\cdot}: M \to M \otimes H$ such that

$$\dot{v}_0 \int_l (gS(\dot{v}_1)) = g * v,$$

where

$$\delta^{\cdot}(v) = \dot{v}_0 \otimes \dot{v}_1. \tag{10}$$

Assume for the moment that for each $h \in H$, there exists an element $F(h)$, such that, for any $g \in H$,

$$\int_l (gS(F(h))) = \int_l (hS(g)). \tag{11}$$

Note that if $F(h)$ exists for an element h, then it is uniquely determined, thanks to the nondegeneracy of the integral. For $v \in V$, choose $h \in H$, such that $h * v = v$, and set

$$\delta^{\cdot}(v) = h_1 * v \otimes F(h_2). \tag{12}$$

LEMMA 2.2. *Assume that the map F, defined by the condition in* (11), *exists. Then the map $\delta^{\cdot}$ defined in* (12) *is a well-defined coaction of H on V, satisfying the condition* (10). *Consequently, the functor $\dot{}$ and $\check{}$ are inverse to each other.*

Proof. Let $h * v = v$. Then, for any $g \in H$,

$$\begin{aligned}\dot{v}_0 \int_l (gS(\dot{v}_1)) &= h_1 * v \int_l (gS(F(h_2))) = h_1 * v \int_l (h_2 S(g)) \\ &= (g * h) * v = g * v.\end{aligned}$$

Conversely, let V be an H-comodule. Then the coaction on $\dot{\check{V}}$ is given as follows. For $v \in V$, let $h \in H$ be such that $v_0 \int_l (hS(v_1)) = v$. Then

$$\dot{\check{\delta}}(v) = v_0 \int_l (h_1 S(v_1)) \otimes F(h_2).$$

We show that $\dot{\check{\delta}} = \delta$. First, notice that applying the coproduct on the equality $v_0 \int_l (hS(v_2)) = v$ we get $v_0 \otimes v_1 = v_0 \otimes v_1 \int_l (hS(v_2))$. Then, using (4), we have, for all $g \in H$,

$$\begin{aligned}v_0 \int_l (h_1 S(v_1)) \int_l (gS(F(h_2))) &= v_0 \int_l (h_1 S(v_1)) \int_l (h_2 S(g)) \\ &= v_0 \int_l (g_2 S(v_1)) \int_l (hS(g_1)) \\ &= v_0 \int_l (gS(v_1)) \int_l (hS(v_2)) \\ &= v_0 \int_l (gS(v_1)),\end{aligned}$$

which implies

$$v_0 \otimes \int_l (h_1 S(v_1)) \otimes F(h_2) = v_0 \otimes v_1.$$

The lemma is proved. ∎

It turns out that the map F can be given in terms of a distinguished group like element, introduced by Radford [19]. In fact, it is shown by Radford that there exists a group like element γ in H, satisfying the condition

$$\int_l (S(g)) = \int_l (g\gamma^{-1}).$$

Set $F(g) = \gamma S^{-2}(g)$. Then we have

$$\int_l (gS(F(h))) = \int_l (gS^{-1}(h)\gamma^{-1}) = \int_l (hS(g)).$$

Composing the operations ˇ and ˆ on a simple comodule M we obtain a new simple comodule $\widehat{\check{M}}$, denoted by $M^\bullet$. The coaction of H on $M^\bullet$ is given by

$$\delta^\bullet(v) = v_0 \int_l (h_1 S(v_1)) \otimes h_2,$$

with h given by condition $v_0 \int_l (hS(v_1)) = v$. Let Γ be the one-dimensional H-comodule, generated by γ. Then it is easy to check that $M^\bullet \cong \Gamma^* \otimes M^{**}$.

THEOREM 2.3. *Let H be a Hopf algebra with integral. Then for any simple comodules M_α, M_β*

$$\dim_k \mathrm{Hom}(\mathcal{J}(M_\alpha), M_\beta^\bullet) = \delta_\beta^\alpha d_\beta^2,$$

where d_β^2 is the dimension over k of $\mathcal{D}_\beta = \mathrm{End}^H(M_\beta)$.

Proof. We first assume that k is algebraically closed. Let M_α be a simple right H-comodule. Then $\check{M}_\alpha$ is a simple left $\check{H}$-module. The action of $\check{H}$ on $\check{M}_\alpha$ induces and $\check{H}$-module homomorphism $\pi\colon \check{H} \to \check{M}_\alpha \otimes (\check{M}_\alpha^*)$, where $\check{M}_\alpha \otimes (\check{M}_\alpha^*) := \check{M}_\alpha^{\oplus \dim_k M_\alpha}$ as $\check{H}$-modules. The isomorphism (9) shows that π is a homomorphism of H-comodules

$$\pi\colon H \to M^\bullet \otimes (M^\bullet),$$

since $(M^\bullet)\check{} = \check{M}$.

On the other hand, H decomposes into the direct sum of its indecomposable injective subcomodules as in Section 1.6. For $h \in H \leftharpoonup e_{\beta,j}$, i.e., $h = e_{\beta,j}(g_1)g_2$ for some $g \in H$ ($e_{\beta,i}$ is a linear functional as in Section 1.6), and for $v \in \check{M}_\alpha$, we have

$$h * v = v_0 e_{\beta,j}(g_1) \int_l (g_2 S(v_1)) = v_0 \int_l (gS(v_1)) e_{\beta,i}(v_2).$$

Thus, if $\alpha \neq \beta$, $h * M_\alpha = 0$, therefore, $\pi(h) = 0$. In other words, the restriction of π on $\mathcal{J}(M_\beta)$ is zero, for $\beta \neq \alpha$. Since π itself is non-zero, its restriction of $\mathcal{J}(M_\alpha)$ should be non-zero. That is,

$$\mathrm{Hom}^H(\mathcal{J}(M_\alpha), M_\alpha^\bullet) \neq 0. \tag{13}$$

According to 1.4,

$$\dim_k(\mathrm{Hom}^H(H, M_\alpha^\bullet)) = \dim_k M_\alpha^\bullet. \tag{14}$$

Since H contains precisely $m_\alpha = \dim_k M_\alpha$ copies of $\mathcal{J}(M_\alpha)$, we conclude that

$$\dim_k \mathrm{Hom}^H(\mathcal{J}(M_\alpha), M_\alpha^\bullet) = 1, \tag{15}$$

$$\dim_k \mathrm{Hom}^H(\mathcal{J}(M_\beta), M_\alpha^\bullet) = 0 \text{ if } \alpha \neq \beta. \tag{16}$$

Consider now the general case. Let $\bar{k}$ be the algebraic closure of k. We have $\mathcal{D}_\beta = \mathrm{End}^H(M_\beta)$ splits over $\bar{k}$: $\mathcal{D}_\beta \otimes_k \bar{k} \cong M_{\bar{k}}(d_\beta)$. For the extension $\overline{H} := H \otimes_k \bar{k}$, the comodule $\overline{M_\beta} := M_\beta \otimes_k \bar{k}$ decomposes into a direct sum of d_β exemplars of a simple $\overline{H}$-comodule M'_β. Since $\overline{\mathcal{J}(M_\beta)} := \mathcal{J}(M_\beta) \otimes_k \bar{k}$ remains a direct summand of $\overline{H}$, it is an injective $\overline{H}$-comodule. Therefore $\overline{\mathcal{J}(M_\beta)}$ is a direct sum of d_β exemplars of $\mathcal{J}(M'_\beta)$. Since, for $\alpha \neq \beta$,

$$\mathrm{Hom}_{\overline{H}}(\mathcal{J}(M'_\alpha), M^{\bullet\prime}_\beta) = 0,$$

we have

$$\mathrm{Hom}_H(\mathcal{J}(M_\alpha), M^\bullet_\beta) = 0.$$

Therefore, by virtue of Eq. (14) (which is valid on any field),

$$\dim_k \mathrm{Hom}_H(\mathcal{J}(M_\alpha)^{\oplus m_\alpha}, M^\bullet_\alpha) = \dim_k M_\alpha = d_\alpha^2 m_\alpha.$$

Consequently

$$\dim_k \mathrm{Hom}_H(\mathcal{J}(M_\alpha), M^\bullet_\alpha) = d_\alpha^2.$$

The theorem is proved. ∎

COROLLARY 2.4. *Let H be a Hopf algebra with integral. Then for any simple comodule M*

$$\mathcal{J}((M^\bullet)^*) \cong \mathcal{J}(M)^*.$$

3. SPLITTING AND NON-SPLITTING COMODULES

Let H be a Hopf algebra and let M be a simple comodule over H. M is called a splitting comodule, or a typical comodule, if $M = \mathcal{J}(M)$. Since $\mathcal{J}(M)$ is injective and hence projective, we see that M splits in any comodule. This explains the term "splitting." The term "typical" was used by Kac for irreducible representations of Lie superalgebras with the above property [13].

By virtue of conditions in Section 1.3, if a Hopf algebra possesses a splitting comodule then it possesses a non-zero integral. The converse statement is not true. A Hopf algebra with integral need not necessarily possess a splitting comodule, e.g., the Sweedler, four-dimensional Hopf algebra [22]. The aim of this section is to give a criterion for a simple comodule to be typical.

Let M now be a simple H-comodule. Consider the $\check{H}$-module $\check{M}$ given by $h \circ v = v_0 \int_l (hS(v_1))$; it is simple on $\check{H}$. Hence, for every $g \in M$, the homomorphism ϕ_g: $\check{H} \to \check{M}$, $h \to h \circ g$ is surjective. By virtue of (9), this map is also the homomorphism of H-comodules $H \to M^\bullet$. Thus, we have associated to each element g from M a comodule homomorphism $\phi_g : H \to M^\bullet$, given by

$$\phi_g(h) = g_1 \int_l (hS(g_2)).$$

Theorem 2.3 ensures that the restriction of ϕ_g on $\mathcal{J}(\mathcal{C}f(M))$ remains surjective.

THEOREM 3.1. *Let M be a simple H-comodule. Then M is a splitting comodule if and only if the bilinear form $b(x, y) := \int_l (xS(y))$ does not identically vanish on $\mathcal{C}f(M)$. In this case b is non-degenerate on $\mathcal{C}f(M)$.*

Proof. As an H-comodule, $\mathcal{C}f(M)$ decomposes into a direct sum of copies of M. Fix such a decomposition. Assume that there exist $g, h \in \mathcal{C}f(M)$, such that $\int_l (hS(g)) \neq 0$. Since $\mathcal{C}f(M)$ is a direct sum of copies of M, we can assume that $g \in M \subset \mathcal{C}f(M)$ and h belongs to another subcomodule of $\mathcal{C}f(M)$, isomorphic to M. The morphism ϕ_g, defined above, then the M onto $M^\bullet$. Consequently, $M \cong M^\bullet$; hence, by Theorem 2.3, M splits in $\mathcal{J}(M)$. Therefore, $M = \mathcal{J}(M)$; that is, M is splitting.

Assume that M is splitting, then $\mathcal{J}(\mathcal{C}f(M)) = \mathcal{C}f(M)$. Thus, the restriction of ϕ_g on $\mathcal{C}f(M)$ is surjective; hence there exists an h from $\mathcal{C}f(M)$ such that $\phi_g(h) \neq 0$. The existence of such an h means that the bilinear form $b(x, y) := \int_l (xS(y))$ does not vanish identically on $\mathcal{C}f(M)$. Thus, we have proved the first part of the theorem.

Assume now that $b(x, y)$ does not vanish on $\mathcal{C}f(M)$. We prove that it is non-degenerate on $\mathcal{C}f(M)$. Consider an element $g \in M \subset \mathcal{C}f(M) \subset H$. Let $\langle g \rangle$ be the comodule generated by g. According to the proof of Theorem 2.3, $\langle g \rangle = \mathcal{C}f(M) * g$. Hence there exists an $h \in \mathcal{C}f(M)$ such that $h * g = \int_l (gS(h_1))h_2 \neq 0$.

On the other hand, if M is splitting then $M \cong M^\bullet$. Hence $\mathcal{C}f(M) \circ M = M$. Embedding M in $\mathcal{C}f(M)$, we can choose an element $f \in M$ such that $\varepsilon(f) \neq 0$. There exists an $h \in \mathcal{C}f(M)$ such that $f = h \circ g = g_0 \int_l (hS(g_1))$. We have $0 \neq \varepsilon(f) = \int_l (hS(g))$.

The proof is complete. ∎

Thus, we see that if M is splitting then $M \cong M^\bullet \cong \Gamma \otimes M^{**}$. If it appears that $M \cong M^{**}$, which is the case when H is coquasitriangular, then we have $M^{**} \cong \Gamma \otimes M^{**}$, which implies $M \cong \Gamma^* \otimes M$. By definition of M^*, there exists a morphism $k \to M \otimes M^{**}$; hence, for any $n \geq 0$, there exists a morphism $\Gamma^{\otimes n} \to M \otimes M^*$. Since M has a finite dimension over k, there exists n, such that $\Gamma^{\otimes n} \cong k$; in other words, γ is unipotent. If H is an integral domain, this implies also that γ is equal to 1.

COROLLARY 3.2. *Assume that the Hopf algebra H is an integral domain and possesses a splitting comodule isomorphic to its double dual. Then the spaces of left and right integrals on H coincide.*

The following lemma will be used in the next section.

LEMMA 3.3. *Let M be a simple H-comodule. Then M is splitting iff M^* is splitting.*

Proof. In fact, if M or M^* is splitting then H has an integral; hence the antipode is bijective, therefore Lemma 1.7 (ii) applies. ∎

We now come to the question about how to define a simple comodule from its injective envelope. For any $g \in N$, and H-comodule, we can define a homomorphism $\phi_g: \check{H} \to N$ of H-comodules, $h \mapsto h * g = g_0 \int_l (g_1 S(h))$. Thus, we obtain a linear correspondence $g \mapsto \phi_g$, which is injective by the non-degeneracy of the integral. On the other hand, according to Section 1.6, the linear space of homomorphism from $H \to N$ has the dimension equal to $\dim_k N$. Therefore, the correspondence above is one to one. In other words, each homomorphism $H \to N$ has the form ϕ_g for some $g \in N$.

Consider a projection given in the first section by an idempotent $e_{\alpha,i}$. Any such homomorphism can be given in terms of an element g_{α_i} in H. Denote this homomorphism by $\phi_{g_{\alpha_i}}$. $\phi_{g_{\alpha_i}}$ being surjective means that g generates $\mathcal{J}(M)$ as H-comodules.

PROPOSITION 3.4. *Let H be a Hopf algebra with integral. Then*

(1) *Any homomorphism from H to a comodule N can be given as ψ_g: $h \mapsto g_0 \int_l (g_1 S(h))$. In particular, the primitive idempotent $e_{\alpha,i}$ in H^* defining the injective envelope $\mathcal{J}(M_\alpha)$ of the simple comodule M_α can be given in terms of an element $g_{\alpha,i}$ as follows $e_{\alpha,i}(h) = \int_l (hS(g_{\alpha,i}))$* (*cf*. [1, Lemma 3.2]).

(2) *Indecomposable injective comodules are cyclic.*

Fix an embedding of $\mathcal{J}(M_\alpha)$ into H. Then there exists a unique projection of H on $\mathcal{J}(M_\alpha)$ mapping $\mathcal{J}(M_\alpha)$ identically into itself. This homomorphism, according to the proposition above, is given in terms of a unique element $g_\alpha \in \mathcal{J}(M_\alpha)$. Thus, we have

COROLLARY 3.5. *Let H be a Hopf algebra with integral. Fix a decomposition $H = \mathcal{J}(M_\alpha) \oplus N$ as H-comodules. Then there exists a unique element g_α in $\mathcal{J}(M_\alpha)$, generating this comodule and such that $h * g$ is equal to h for all $h \in \mathcal{J}(M_\alpha)$ and is equal to 0 for h in N.*

Notice that the choice of g_α depends on the way of embedding $\mathcal{J}(M_\alpha)$ into H. If, however, we consider $\mathcal{J}(\mathcal{C}f(M_\alpha))$, for which there exists a unique embedding, repeating the above process we obtain a unique element G_α in $\mathcal{J}(\mathcal{C}f(M_\alpha))$ with analogous property.

From the general theory of socles, a comodule V is coideal (i.e., can be embedded into H) if and only if its socle is coideal.

COROLLARY 3.6. *A comodule V which is finite-dimensional over k is coideal if and only if V^* is cyclic.*

Proof. If V is coideal then $\mathcal{J}(V)$ is a direct summand of H; hence $\mathcal{J}(V)$ is a direct summand of H, therefore cyclic. Since V^* is a quotient of $\mathcal{J}(V)^*$; it is cyclic too.

Conversely, if V is cyclic then there exists a finite-dimensional (over k) direct summand of H with a surjective map on V; hence V^* can be embedded in its dual which is also a direct summand of H, and thus is injective. ∎

Simple comodules can be recovered from their injective envelopes as follows.

PROPOSITION 3.7. *Let M_α be a simple comodule. Then $\mathcal{J}(M_\alpha)$ contains a unique maximal subcomodule, which can be characterized as the union of the kernels of non-invertible maps $\mathcal{J}(M_\alpha) \to \mathcal{J}(M_\alpha^\bullet)$ of the form $h \mapsto g_1 \int_l (hS(g_2))$.*

Proof. The maximal subcomodule of $\mathcal{J}(M_\alpha)$ is given as the kernel of $\mathcal{J}(M_\alpha) \to M_\alpha^\bullet$. Thus it can be considered as the union of the kernels of non-invertible homomorphisms f: $\mathcal{J}(M_\alpha) \to \mathcal{J}(M_\alpha^\bullet)$, since any such non-zero map should have the image containing $M_\alpha^\bullet$. Embed $\mathcal{J}(M_\alpha)$ in H and fix a canonical generator g_α. Then

$$f(a) = f(a * g_\alpha) = a \circ f(g_\alpha).$$

In other words, f is given in terms of $h = f(g_\alpha)$:

$$f(h) = h \circ g = g_1 \int_l (h(S(g_2)).$$

The proof is complete. ∎

4. SIMPLE REPRESENTATIONS OF QUANTUM GROUPS OF TYPE $A_{0|0}$

Quantum groups of type $A_{0|0}$ are Hopf algebras, given in terms of Hecke symmetries of birank $(1, 1)$. It is shown in [9] that on such a Hopf algebra, integrals exist. Applying the criteria obtained in the previous section, we study simple comodules of these Hopf algebras. Theorem 4.1 classifies all simple comodules. As a consequence, the Hecke symmetries are also classified.

We begin with some definitions. Let V be a finite-dimensional vector space over k, a field of characteristic zero. An operator R: $V \otimes V \to V \otimes V$ is called a Hecke symmetry if R satisfies the Yang–Baxter equation

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R),$$

the Hecke equation

$$(R - \mathrm{id})(R - q \cdot \mathrm{id}) = 0, \;\; q \neq 0,$$

and is closed; that is, the operator P: $V^* \otimes V \to V \otimes V^*$, called half dual (half-adjoint) to R, given by

$$P = (\mathrm{ev}_V \otimes \mathrm{id}_{V \otimes V^*})(\mathrm{id}_{V^*} \otimes R \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^* \otimes V} \otimes \mathrm{db}_V),$$

is invertible. We shall also assume that q is not a root of unity of degree greater than 1.

Being given a Hecke symmetry, one can define the associated quantum symmetric and antisymmetric tensor algebras as factor algebras of the tensor algebra over V by the ideals, generated by $\mathrm{Im}(R - q \cdot \mathrm{id})$ and $\mathrm{Im}(R + \mathrm{id})$, respectively. It is shown that the Poincaré series of these algebras, i.e., the formal power series with coefficients being dimensions of homogeneous components of these algebras, are rational functions having negative roots and positive poles [11].

A Hecke symmetry R is said to have birank $(1, 1)$ if the Poincaré series of the associated quantum symmetric tensor algebra has one pole and one root, i.e., is of the form $(1 + at)(1 - bt)^{-1}$, $a, b > 0$.

The quantum group (quantum semigroup) associated to R is defined to be the Hopf algebra (bialgebra) universally coacting on the mentioned above quantum symmetric and antisymmetric tensor algebras [17] (see also [8]). They are denoted by H and E, respectively. The Yang–Baxter equation for R implies that H and E are coquasitriangular Hopf algebra and bialgebra, respectively. If R has birank $(1, 1)$, the associated quantum group is called quantum group of type $A_{0|0}$. Simple E-comodules can be labeled by hook-partitions of the form $(m, 1^n)$, $m \geq 1, n \geq 0$ and the trivial partition (0) [8, Theorem 2.3.2]. For simplicity we shall use the pair (m, n) to denote the partition $(m, 1^n)$ and the pair $(0, 0)$ to denote the trivial partition. The endomorphisms ring of a simple E-comodule is isomorphic to k. On the other hand, simple E-comodules are also simple as H-comodules with the natural action induced from the inclusion $E \to H$ [8, Theorem 2.3.5].

From now on, for simplicity we shall use a dot "$\cdot$" to denote the tensor product and a plus sign "+" to denote the direct sum, thus V^n will mean $V^{\otimes n}$ and $n \cdot V$ will mean $V^{\oplus n}$. We shall also use the equal sign "=" to denote an isomorphism.

Simple E-comodules $\mathrm{I}_{m,n}$, associated to pairs (m, n), $m \geq 1, n \geq 0$, are given by the following rule. $\mathrm{I}_{n,0} = S_n$ is the nth component of the quantum symmetric tensor algebra over V, $\mathrm{I}_{1,n-1} = \Lambda_n$ is the nth component of the quantum antisymmetric tensor algebra over V, $n \geq 1$, $\mathrm{I}_{0,0} := k$ and the following decomposition rules are satisfied

$$\mathrm{I}_{p,\,q}\cdot\mathrm{I}_{m,\,n}=\mathrm{I}_{m+p,\,n+q}+\mathrm{I}_{m+p-1,\,n+q+1},\tag{17}$$

for $m,\,p\geq 1,\,n,\,q\geq 0$. Particularly, we have $V=\mathrm{I}_{1,\,0}$, $V^*=\mathrm{I}_{-1,\,0}$.

Our aim is to associate to each pair (m,n) of integers a simple H-comodule and show that they furnish all simple H-comodules. Namely, we shall prove

THEOREM 4.1. *Irreducible representations of a quantum group of type $A_{0|0}$ are classified by pairs (m,n) of integers with the following properties*:

(1) $\mathrm{I}_{m,\,0}$ *is the nth symmetric tensor*, $\mathrm{I}_{1,\,n-1}$ *is the nth antisymmetric tensor*, $\mathrm{I}_{0,\,0}=k$, $\mathrm{I}^*_{m,\,n}=\mathrm{I}_{-m,\,-n}$, $\mathrm{I}_{1,\,-1}$ *is the super determinant*. $\mathrm{I}_{m,\,n}$ *is splitting iff* $m+n\neq 0$.

(2) *The tensor product of simple comodules obeys the following decomposition rules*:

(a) *for any integers* m,n

$$\mathrm{I}_{m,\,n}\cdot\mathrm{I}_{-1,\,1}=\mathrm{I}_{m-1,\,n+1},$$

(b) *for any* $m>n>0$,

$$\mathrm{I}_{m,\,0}\cdot\mathrm{I}_{n,\,0}=\mathrm{I}_{m+n,\,0}+\mathrm{I}_{m+n-1,\,1}$$
$$\mathrm{I}_{m,\,0}\cdot\mathrm{I}_{-n,\,0}=\mathrm{I}_{m-n,\,0}+\mathrm{I}_{m-n+1,\,-1},$$

(c) *for* $m\neq 0$, *the product* $\mathrm{I}_{m,\,0}\cdot\mathrm{I}_{-m,\,0}=\mathrm{I}_{1,\,0}\cdot\mathrm{I}_{-1,\,0}$ *is injective and indecomposable. It contains two exemplars of* k *and the comodules* $\mathrm{I}_{1,\,-1}$ *and* $\mathrm{I}_{-1,\,1}$ *in its decomposition series*.

Proof. There will be several steps in the proof. In the first step, we shall show the existence of the integral on H.

It is proved in [9, Theorem 3.2], that if $\mathrm{rank}_q R=0$, where $\mathrm{rank}_q R$ is the trace of the half-dual operator P, then H_R possesses an integral. Thus, in order to apply the results of the previous section, we have to show that $\mathrm{rank}_q R=0$. To do this we consider the Koszul complex of the second type introduced by Manin [16] (see also [7, 10, 15]). It is shown that, if $\mathrm{rank}_q R\neq -[k-l]_q$, then the complex

$$\cdots\longrightarrow\Lambda_k\cdot S^*_l\xrightarrow{d_{k,\,l}}\Lambda_{k+1}\cdot S^*_{l+1}\xrightarrow{d_{k+1,\,l+1}}\Lambda_{k+2}\cdot S^*_{l+2}\longrightarrow\cdots$$

with the differential induced from the dual basis map $\mathrm{ev}:k\to V\cdot V^*$, is exact [7, 15].

Notice that, according to (17), (6), and (7), for $m,\,p\geq 1$ and $n,\,q\geq 0$,

$$\mathrm{Hom}^H(\mathrm{I}_{p,\,q},\mathrm{I}_{m+p,\,n+q}\cdot\mathrm{I}^*_{m,\,n})=k$$
$$\mathrm{End}^H(\mathrm{I}_{m+p,\,n+q}\cdot\mathrm{I}^*_{m,\,n})=\mathrm{End}^H(\mathrm{I}_{m+p,\,n+q}\cdot\mathrm{I}_{m,\,n})=2\cdot k.$$

Therefore, denoting $\mathrm{I}_{-m,\,-n} := \mathrm{I}^*_{m,\,n}$, we have, for $p, q \geq 1$, $m \geq 1$, $n \geq 0$,

$$\mathrm{I}_{p,\,q} + \mathrm{I}_{p+1,\,q-1} \subset \mathrm{I}_{m+p,\,n+q} \cdot \mathrm{I}_{-m,\,-n}, \tag{18}$$

and $\mathrm{I}_{m+p,\,n+q} \cdot \mathrm{I}_{-m,\,-n}$ does not contain any other simple E-comodule.

Assume that $\mathrm{rank}_q R \neq 0$. Then the complex

$$0 \longrightarrow k \overset{d_{00}}{\longrightarrow} \mathrm{I}_{1,\,0} \cdot \mathrm{I}_{-1,\,0} \overset{d_{11}}{\longrightarrow} \mathrm{I}_{1,\,1} \cdot \mathrm{I}_{-2,\,0} \overset{d_{11}}{\longrightarrow} \cdots \tag{19}$$

is exact. We have, for $n > m \geq 1$,

$$\begin{aligned} \mathrm{I}_{n,\,1} \cdot \mathrm{I}_{1,\,m-1} \cdot \mathrm{I}_{-m,\,0} &= (\mathrm{I}_{n+1,\,m} + \mathrm{I}_{n,\,m+1}) \cdot \mathrm{I}_{-m,\,0} \\ &\supset 2 \cdot \mathrm{I}_{n-m+1,\,m} + \mathrm{I}_{n-m,\,m+1} + \mathrm{I}_{n-m+2,\,m-1}. \end{aligned} \tag{20}$$

Thus, multiplying (19) with $\mathrm{I}_{n,\,1}$, we have a diagram

$$\begin{array}{ccccccc} \mathrm{I}_{n,1} & & 2\cdot\mathrm{I}_{n,1}+\mathrm{I}_{n-1,2}+\mathrm{I}_{n+1,0} & & 2\cdot\mathrm{I}_{n-1,2}+\mathrm{I}_{n-2,3}+\mathrm{I}_{n,1} & \cdots \\ \| & & \cap & & \cap & \\ 0\longrightarrow \mathrm{I}_{n,1} & \longrightarrow & \mathrm{I}_{n,1}\cdot\mathrm{I}_{1,0}\cdot\mathrm{I}_{-1,0} & \longrightarrow & \mathrm{I}_{n,1}\cdot\mathrm{I}_{1,1}\cdot\mathrm{I}_{-2,0}\longrightarrow & \cdots. \end{array} \tag{21}$$

The exactness of the lower complex and the remark following (18) imply that $S_{n+1} = \mathrm{I}_{n+1,\,0} = 0$, contradicting the assumption on the Poincaré series. Thus we have $\mathrm{rank}_q R = 0$.

As a consequence, the Hopf algebra H possesses an integral and the formula for the integral in [9, Sect. 5] implies that $\mathrm{I}_{m,\,n}$, and hence $\mathrm{I}_{-m,\,-n}$, $m \geq 1$, $n \geq 0$, are all splitting, except for $\mathrm{I}_{0,\,0} = k$. Therefore, by means of the two isomorphisms preceding (18), the inclusion in (18) is in fact an isomorphism: for $p, q, m \geq 1, n \geq 0$,

$$\mathrm{I}_{m+p,\,n+q} \cdot \mathrm{I}_{-m,\,-n} = \mathrm{I}_{p,\,q} + \mathrm{I}_{p+1,\,q-1}. \tag{22}$$

The second step is to define the comodules $\mathrm{I}_{-1,\,1}$ and $\mathrm{I}_{1,\,-1}$.

Consider the sequence (19). Since $\mathrm{rank}_q R = 0$, $\mathrm{I}_{1,\,0} \cdot \mathrm{I}_{-1,\,0} = V \cdot V^*$ contains two exemplars of k in its composition series but only one as subcomodule. Let $M := (V \cdot V^*)/k$, then the homomorphism $V \cdot V^* \to k$ factorizes through k to a homomorphism $M \to k$. Dualizing this we get a sequence $k \to M^* \to V \cdot V^*$. Since

$$\mathrm{Hom}^H(k, \mathrm{I}_{1,\,1} \cdot \mathrm{I}_{-2,\,0}) \cong \mathrm{Hom}^H(\mathrm{I}_{2,\,0}, \mathrm{I}_{1,\,1}) = 0;$$

that is, k cannot be a subcomodule of $\mathrm{I}_{1,\,1} \cdot \mathrm{I}_{-2,\,0}$, $\mathrm{Im} d_{1,\,1} \neq k$. From (21), we see that $\mathrm{Ker}\, d_{2,\,2} \supsetneqq k$. Let $N := (V \cdot V^*)/\mathrm{Ker}\, d_{2,\,2}$. Then N is a factor comodule of M, which is different from k and M. Therefore $V \cdot V^*$ contains at least four simple comodules in its composition series. There cannot be more than four modules, as on the left-hand side of (20); there are four simple comodules. Denote by A and B the two simple subcomodules, that are different from k. Since $V \cdot V^*$ is self dual, either A and B are both self dual or B is dual to A.

Using (22), we have

$$\mathrm{I}_{2,1} \cdot \mathrm{I}_{-1,0} \cdot \mathrm{I}_{1,0} = 2 \cdot \mathrm{I}_{2,1} + \mathrm{I}_{1,2} + \mathrm{I}_{3,0}.$$

Thus, we can assume that $\mathrm{I}_{2,1} \cdot A = \mathrm{I}_{1,2}$ and $\mathrm{I}_{2,1} \cdot B = \mathrm{I}_{3,0}$. Using (double) induction we can easily show that

$$\mathrm{I}_{m,n} \cdot A = \mathrm{I}_{m-1,n+1} \quad \mathrm{I}_{m,n} \cdot B = \mathrm{I}_{m+1,n-1}, \tag{23}$$

for all $m \geq 2, n \geq 1$. Using the fact, that $M \cdot M^*$ contains k as a subcomodule, for any comodule M, we deduce that $A = B^*$ and $A \cdot B = k$. Define $\mathrm{I}_{1,-1} := B$, $\mathrm{I}_{-1,1} := A$. Thus

$$\mathrm{I}_{1,-1} \cdot \mathrm{I}_{-1,1} = \mathrm{I}_0, \tag{24}$$

$$\mathrm{I}_{m,n} \cdot \mathrm{I}_{1,-1} = \mathrm{I}_{m+1,n-1}, \tag{25}$$

$$\mathrm{I}_{m,n} \cdot \mathrm{I}_{-1,1} = \mathrm{I}_{m-1,n+1}, \tag{26}$$

for all $m \geq 2, n \geq 1$. Dualizing these equalities, we obtain

$$\mathrm{I}_{-m,-n} \cdot \mathrm{I}_{1,-1} = \mathrm{I}_{-m+1,-n-1} \tag{27}$$

$$\mathrm{I}_{-m,-n} \cdot \mathrm{I}_{-1,1} = \mathrm{I}_{-m-1,-n+1}, \tag{28}$$

for all $m \geq 2, n \geq 1$.

Consider now

$$\begin{aligned}\mathrm{I}_{1,1} \cdot \mathrm{I}_{1,0} \cdot \mathrm{I}_{-1,0} &= (\mathrm{I}_{2,1} + \mathrm{I}_{1,2}) \cdot \mathrm{I}_{-1,0} \\ &= \mathrm{I}_{1,1} + \mathrm{I}_{2,0} + \mathrm{I}_{1,2} \cdot \mathrm{I}_{-1,0}.\end{aligned}$$

The left-hand side contains simple comodules $\mathrm{I}_{1,1} \cdot \mathrm{I}_{1,-1}$ and $\mathrm{I}_{1,1} \cdot \mathrm{I}_{-1,1}$. Using previous results, we have $\mathrm{Hom}(\mathrm{I}_{1,1} \cdot \mathrm{I}_{1,-1}, \mathrm{I}_{1,1}) = 0$, $\mathrm{Hom}(\mathrm{I}_{1,1} \cdot \mathrm{I}_{1,-1}, \mathrm{I}_{1,2} \cdot \mathrm{I}_{-1,0}) = 0$. We therefore conclude that

$$\mathrm{I}_{1,1} \cdot \mathrm{I}_{1,-1} = \mathrm{I}_{2,0},$$

and thus

$$\mathrm{I}_{2,0} \cdot \mathrm{I}_{-1,1} = \mathrm{I}_{1,1}.$$

We are now at the stage to associate to each pair (m, n) of integers a simple comodule $\mathrm{I}_{m,n}$. Note that for $m \neq 1, n \geq 0$ or $m \leq 1, n \leq 0$, we have already defined $\mathrm{I}_{m,n}$. We call $s(m,n) := m + n$ the total degree of the pair (m, n). Thus, there can be three possibilities: $s(m,n) > 0, < 0$, or $= 0$. If $s(m,n) = 0$, i.e., $m = -n$, set

$$\mathrm{I}_{m,-m} := \mathrm{I}_{1,-1}^m.$$

If $s(m, n) \neq 0$, set

$$\mathrm{I}_{m,n} := \mathrm{I}_{m+n,0} \cdot \mathrm{I}_{-n,n}.$$

Using (24–28), it is easy to see that the newly defined comodules are compatible with the predefined comodules and that these comodules are all simple. We want to find the formula for the tensor product of these comodules and deduce from this formula that these comodules furnish all simple H-comodules.

Let (m, n) and (p, q) be pairs of integers. Our aim is to decompose $\mathrm{I}_{m,n} \cdot \mathrm{I}_{p,q}$—the main role played here is the total degree. There can be three possibilities

(1) either $m+n$ or $p+q$ is equal to zero;

(2) $m+n$ and $p+q$ are both different from zero but their sum is zero;

(3) $m+n$ and $p+q$ and $m+n+p+q$ are all different from zero.

1. If $m+n=0$ then $\mathrm{I}_{m,n} = \mathrm{I}^m_{1,-1}$; hence

$$\mathrm{I}_{m,-m} \cdot \mathrm{I}_{p,q} = \mathrm{I}_{p+m,q-m}. \tag{29}$$

2. If $m+n+p+q=0$ and $m+n \neq 0$, using (24–28), we can assume $n = p = 0$. Thus $m = -p$. We claim that

$$\mathrm{I}_{m,0} \cdot \mathrm{I}_{-m,0} = \mathrm{I}_{1,0} \cdot \mathrm{I}_{-1,0}. \tag{30}$$

Indeed, $\mathrm{I}^*_{m,0} = \mathrm{I}_{-m,0}$; hence $\mathrm{I}_{m,0} \cdot \mathrm{I}_{-m,0}$, contains k as subcomodule. Moreover, this comodule is injective. On the other hand, $\mathrm{I}_{1,0} \cdot \mathrm{I}_{-1,0}$ is the injective envelope of k; therefore (cf. [6]) it is a subcomodule of $\mathrm{I}_{m,0} \cdot \mathrm{I}_{-m,0}$, $\forall m \geq 0$. Multiplying these comodules with $\mathrm{I}_{m,1}$ we get the same comodule. Whence we conclude (30).

3. If $m+n$, $p+q$, $m+n+p+q$ are all non-zero, dualizing if necessary, we can assume $m+n+p+q > 0$. Using (24–28), we can assume $n = q = 0$. Assume $m > p$; thus $m > 0$. One is led to compute $\mathrm{I}_{m,0} \cdot \mathrm{I}_{p,0}$. If $p > 0$, the formula is already known (cf. 17 and 18). Assume $p < 0$ and set $k = -p$, then $k > 0$ and $m > k$. We consider two cases: $m - k \geq 2$ and $m - k = 1$. If $m - k \geq 2$, then, according to (24–28),

$$\begin{aligned}\mathrm{I}_{m,0} \cdot \mathrm{I}_{-k,0} &= \mathrm{I}_{1,-1} \cdot \mathrm{I}_{m-1,1} \cdot \mathrm{I}_{-k,0} \\ &= \mathrm{I}_{1,-1} \cdot (\mathrm{I}_{m-k-1,1} + \mathrm{I}_{m-k,0}) \\ &= \mathrm{I}_{m-k,0} + \mathrm{I}_{m-k+1,-1}.\end{aligned} \tag{31}$$

In the case $m - k = 1$, we show that

$$\mathrm{I}_{m,0} \cdot \mathrm{I}_{-m+1,0} = \mathrm{I}_{1,0} + \mathrm{I}_{2,-1}. \tag{32}$$

We have

$$\begin{aligned}\mathrm{Hom}(\mathrm{I}_{1,0}, \mathrm{I}_{m,0} \cdot \mathrm{I}_{-m+1,0}) &= \mathrm{Hom}(\mathrm{I}_{1,0} \cdot \mathrm{I}_{m-1,0}, \mathrm{I}_{m,0}) \\ &= k.\end{aligned}$$

Remember that $\mathrm{I}_{2,-1} = \mathrm{I}_{1,-1} \cdot \mathrm{I}_{1,0}$ and that $\mathrm{I}_{2,-1}$ is also simple; hence

$$\begin{aligned}\mathrm{Hom}(\mathrm{I}_{2,-1}, \mathrm{I}_{m,0} \cdot \mathrm{I}_{-m+1,0}) &= \mathrm{Hom}(\mathrm{I}_{1,-1} \cdot \mathrm{I}_{1,0}, \mathrm{I}_{m,0} \cdot \mathrm{I}_{-m+1,0}) \\ &= \mathrm{Hom}(\mathrm{I}_{1,0} \cdot \mathrm{I}_{m-1,0}, \mathrm{I}_{-1,1} \cdot \mathrm{I}_{m,0}) \\ &= k.\end{aligned}$$

Thus $\mathrm{I}_{m,0} \cdot \mathrm{I}_{-m+1,0}$ contains $\mathrm{I}_{1,0}$ and $\mathrm{I}_{2,-1}$ as subcomodules. On the other hand, multiplying both sides of (32) with $\mathrm{I}_{m+2,1}$, we get an equality. Therefore (32) is proven. The proof is complete. ∎

The classification obtained above also allows us to classify Hecke symmetries of birank (1, 1). The crucial point here is to compute the dimension of simple comodules. Since $\mathrm{I}_{1,-1} \cdot \mathrm{I}_{-1,1} = \mathrm{I}_{0,0} = k$, $\mathrm{I}_{1,-1}$ is one-dimensional. On the other hand, assuming that the Poincaré series of the quantum anti-symmetric algebra Λ is $(1 + at)(1 - bt)^{-1}$ with $a, b > 0$, we can compute the dimension of comodules $\mathrm{I}_{m,n}$, $m \geq 1, n \geq 0$,

$$\dim_k \mathrm{I}_{m,n} = a^m b^n + a^{m-1} b^{n+1}.$$

According to (25), we have $a = b$. On the other hand, computing the dimension of $\mathrm{I}_{1,0} \cdot \mathrm{I}_{-1,0}$ in two ways we obtain $a + b = 4$. Therefore $a = b = 2$; that is, $\dim_k V = 2$. That means, a Hecke symmetry of birank (1, 1) should be defined on a vector space of dimension 2. According to the classification result of Hietarinta [12] on solutions to the Yang–Baxter equation of dimension 2, there are only two families of such operators. The first one is two-paramenteric, found by Manin [18]; the second one is one-paramenteric, found by Takeuchi and Tambara [23].

ACKNOWLEDGMENTS

Parts of the work were done during the author's stay at the Max-Planck-Institut für Mathematik, Bonn, Germany, and the Mathematical Sciences Research Institute, Berkeley, California. The author thanks these institutes for financial support.

REFERENCES

1. M. Beattie, S. Dascalescu, L. Gruenenfelder, and Nastasescu, Finiteness conditions, co-Frobenius Hopf algebras, and quantum groups, *J. Algebra* **200** (1998), 312–333.
2. M. Beattie, S. Dascalescu, and L. Grünenfelder, Constructing pointed Hopf algebras by Ore extensions, *J. Algebra* **225** (2000), 743–770.
3. F. A. Berezin, "Introduction to Superanalysis," Reidel, Dordrecht, 1987.
4. C. Cai and H. Chen, Coactions, smash products, and Hopf modules, *J. Algebra* **167** (1994), 85–99.
5. Yukio Doi, Homological coalgebra, *J. Math. Soc. Japan* **33**, No. 1 (1981), 31–50.
6. J. A. Green, Locally finite representations, *J. Algebra* **41** (1976), 137–171.
7. D. I. Gurevich, Algebraic aspects of the quantum Yang–Baxter equation, *Leningrad Math. J.* **2**, No. 4 (1991), 801–828.
8. Phung Ho Hai, On matrix quantum groups of type A_n, *Internat. J. Math.*, **11**, No. 9 (2000), 1115–1146.
9. Phung Ho Hai, "The Integral on Quantum Super Groups of Type $A_{r|s}$," *Asian J. Math.*, in press; available at `xxx.lanl.gov`.
10. Phung Ho Hai, On the structure of quantum super groups $\mathrm{GL}_q(m|n)$, *J. Algebra* **211** (1999), 363–383.
11. Phung Ho Hai, Poincaré series of quantum spaces associated to Hecke operators, *Acta Math. Vietnam* **24**, No. 2 (1999), 236–246.
12. J. Hietarinta, Solving the two-dimensional constant quantum Yang–Baxter equation, *J. Math. Phys.* **34**, No. 5 (1993), 1725–1732.
13. V. G. Kac, Characters of typical representations of classical Lie superalgebras, *Comm. Algebra* **5**, No. 8 (1977), 889–897.
14. B. I. Lin, Semiperfect coalgebras, *J. Algebra* **49** (1977), 357–373.
15. V. V. Lyubashenko and A. Sudbery, Quantum super groups of $\mathrm{GL}(n|m)$ type: Differential forms, Koszul complexes and Berezinians, *Duke Math. J.* **90** (1977), 1–62.
16. Yu. I. Manin, "Gauge Field Theory and Complex Geometry," Springer-Verlag, Berlin/New York, 1988.
17. Yu. I. Manin, "Quantum Groups and Non-commutative Geometry," GRM, Univ. de Montreal, 1988.
18. Yu. I. Manin, Multiparametric quantum deformation of the general linear supergroups, *Comm. Math. Phys.* **123** (1989), 163–175.
19. D. Radford, Finiteness conditions for a Hopf algebra with non-zero integral, *J. Algebra* **46** (1977), 189–195.
20. Dragoş Ştefan, The uniqueness of integrals (a homological approach), *Comm. Algebra* **23**, No. 5 (1995), 1657–1662.
21. J. B. Sullivan, The uniqueness of integral for Hopf algebras and some existence theorems of integrals for commutative Hopf algebras, *J. Algebra* **19** (1971), 426–440.
22. M. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.
23. M. Takeuchi and D. Tambara, A new one-parameter family of 2×2 quantum matrices, *Hokkaido Math. J.* **XXI**, No. 3 (1992), 409–419. [See also *Proc. Japan. Acad.* **67**, No. 8 (1991), 267–269.]
24. S. L. Woronowicz and Podles, Quantum deformation of Lorentz group, *Comm. Math. Phys.* **130** (1990), 381–431.